An Analytical and Computer Aided Approach to Fourier Analysis, Synthesis, and Transform

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Abstract – Analytical and simulation approaches for an introduction of the Fourier series and transform are presented in an integrated approach. A purely analytical approach is presented first for the Fourier series of a function. The Fourier series and transform are applied to a simple RC low pass filter network illustrating the process. A comparison of simulation and analytical results is attempted along with an explanation of discrepancies.

Keywords: Fourier series, Fourier transform, Simulation, Electric circuits.

INTRODUCTION

In order to enhance the teaching and improve the understanding by the students, of the Fourier series and transform, an analytical and computer aided unified approach is presented. The analytical approach follows the traditional lecture-presentation approach. But this approach is enhanced using a mathematical notepad [1] and an electric circuit simulator [2]. The ability of the last two tools to perform fast calculations and display the resultant output in the form of a table or a graph, greatly attracts the attention of the students, given the short attention span in a long academic period. In addition these computer programs allow further investigation and the performance of “what if” inquiries.

Both computer aided tools can help to comprehend concepts such as the frequency spectrum of a waveform (analysis), and waveform composition (synthesis). Electrical and, mostly, electronic circuits, process non sinusoidal signals. When these signals are periodic they can be expressed in terms of sinusoids, which in turn, can be applied separately at the input of a system. When the input signal is at steady state, the output will be periodic of the same period. To decompose a non sinusoid periodic function into sinusoid components, Fourier series analysis is used. Standard circuit analysis techniques, such as phasor analysis, can be used to analyze a circuit. For the case of linear circuits, superposition can be used to determine the output waveform.

FOURIER ANALYSIS AND SYNTHESIS - THEORY

Fourier analysis is the representation of a function by a sum of components while Fourier synthesis is the composition of a function by its components. The representation by considering the first few number of terms, is approximate, but adequate for many practical applications due to the convergence of the series representation. Joseph Fourier showed that representing a function as a sum of trigonometric components simplifies the study of the problem. Fourier was studying heat propagation in 1912 [3].

Any practical waveform, but a sinusoid, can be expressed as a sum of components. These components have an average or DC component, a first sinusoid or fundamental at the fundamental frequency, and an infinite number of harmonic sinusoids. Together all sinusoid components consist the spectrum of the waveform. The information provided by the spectrum of a waveform is useful in to determine the bandwidth required to transmit adequately a signal. Due to its great importance in signal analysis, the subject of Fourier series and transform is included as early

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as the second semester of a two semester sequence of a course in Electric Circuits [4], and furthermore elaborated in a course of Signal and Systems, [5].

When a steady-state periodic signal is applied to the input of a linear system, the output will be periodic. The period of the input and output signals are the same. In Fourier analysis a periodic signal is decomposed in a sum of components, called harmonics, which are functions that contain trigonometric terms either sinusoidal or cosinusoidal or both. [6]

Mathematically, a periodic function $f(t)$ having fundamental period $T$ and angular frequency $\omega_0 = 2\pi / T$ can be expressed as an infinite summation of basic trigonometric sinusoidal and cosinusoidal terms, in the form

$$
f(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]
$$

A periodic function can be expressed in terms of components if it satisfies Dirichlet conditions:

- $f(t)$ must be single valued,
- $f(t)$ must have a finite number of discontinuities within one period,
- $f(t)$ must have a finite number of maxima and minima within one period, and
- The integral $\int_{t_0}^{t_0+T} |f(t)| dt$ must exist for any $t$.

The periodic functions of physical interest do satisfy Dirichlet conditions.

There are two other equivalent forms of the Fourier series, the amplitude phase cosine and the amplitude phase sine

$$
f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)
$$

$$
f(t) = A_0 + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t + \phi_n)
$$

$A_0$, $B_0$, $F_0$ ils the full cycle average of $f(t)$,

$A_n$, $B_n$, are the amplitude of the nth cosinusoidal or sinusoidal component

$n\omega_0$ is the angular frequency of the nth component, and

$\theta_n$, $\phi_n$ is the phase angle of the nth cosinusoidal or sinusoidal component.

In addition, a signal can be expressed in exponential form as

$$
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{i n\omega_0 t}
$$

The task of Fourier series analysis is to determine the values of the coefficients $a_n$, $b_n$, $A_n$, $B_n$, $F_n$, $\theta_n$, $\phi_n$ in the various expressions of the Fourier series above. Note that physically,
represents the average value of the function.

The Fourier coefficients of the other terms are given by

\[ a_n = \frac{2}{T} \int_0^T f(t) \cos(n \omega_0 t) \, dt \quad n = 1, 2, 3, \ldots \] (6)

\[ b_n = \frac{2}{T} \int_0^T f(t) \sin(n \omega_0 t) \, dt \quad n = 1, 2, 3, \ldots \] (7)

\[ F_n = \frac{1}{T} \int_0^T f(t) \exp(-jn \omega_0 t) \, dt \quad n = \pm 1, \pm 2, \pm 3, \ldots \] (8)

\[ c_n = \sqrt{a_n^2 + b_n^2} = 2 |F_n|, \quad n = 1, 2, 3, \ldots \] (9)

\[ \theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) \equiv \angle F_n, \quad n = 1, 2, 3, \ldots \] (10)

Physically, the Fourier series expansion indicates that, if a signal has a finite energy over an arbitrary interval, then, it can be expanded as a linear combination of an infinite number of harmonically related sinusoids or complex exponential functions. The coefficients are called the Fourier coefficients. The fundamental frequency is equal to the inverse of the fundamental period. If the signal is periodic, the Fourier series representation is valid over \(-\infty < t < +\infty\). If the signal is not periodic, then the Fourier series representation is valid only over the interval \(0 < t < T\). When the coefficients are plotted against the harmonic frequencies, the graph is called the spectrum of the signal. The coefficients, a complex number in general, can be represented in polar form, by amplitude and phase. The graph of the amplitude against the harmonic frequencies is called the amplitude spectrum. The graph of the phase against the harmonic frequencies is called the phase spectrum.

**FOURIER ANALYSIS AND SYNTHESIS – THEORY: EXAMPLE**

Let us consider, as a specific application, a sawtooth wave, normalized for amplitude equal to 1 and period equal to 1. Figure 1 shows a graph of a sawtooth wave, which can be the snapshot of a signal from an oscilloscope. The equation of this specific sawtooth wave is

\[ x(t) = t - \left\lfloor t \right\rfloor = t - \text{floor}(t) \] (11)

![Figure 1. Sawtooth function.](image)
Evaluation of the Fourier coefficient gives:

\[
a_0 = \frac{1}{T} \int_0^T A dt = \frac{A}{2}
\]  
(12)

\[
a_n = \frac{2}{T} \int_0^T A \cos(n\omega_0 t) dt = \frac{2A}{T^2} \left( \frac{\cos(n\omega_0 T)}{(n\omega_0)^2} + \frac{t \sin(n\omega_0 t)}{n\omega_0} \right) \bigg|_0^T = 0
\]  
(13)

All the \(a_n\) terms are zero.

\[
b_n = \frac{2}{T} \int_0^T A \sin(n\omega_0 t) dt = \frac{2A}{T^2} \left( \frac{\sin(n\omega_0 T)}{(n\omega_0)^2} + \frac{t \cos(n\omega_0 t)}{n\omega_0} \right) \bigg|_0^T = -\frac{A}{n\pi}
\]  
(14)

Thus the specific sawtooth wave can be expressed in a sine Fourier series as:

\[
f(t) = A \left( \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\omega_0 t) \right)
\]  
(15)

And, equivalently, as the amplitude phase cosine series in the form:

\[
f(t) = A \left( \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos(n\omega_0 t + 90^\circ) \right)
\]  
(16)

Physically, the above expressions indicate that for the sawtooth function the amplitude of each component is inversely proportional to its harmonic number \(n\). Moreover, all harmonics have the same phase angle.

**Fourier Analysis and Synthesis - Simulation**

The free Cadence OrCAD (OrCAD PSPICE) version, a commercial product derived from the Berkley SPICE [7] is used. Fourier series coefficients of a waveform are computed from the last complete cycle of a transient analysis. Transient analysis must precede Fourier series analysis. The TSTOP option of the transient analysis must be set to at least equal to \(T\), one period of the waveform. For the special and of practical importance, case of the Fourier series coefficients of a waveform in steady state, the TSTOP option of the transient analysis should be set to include a number of cycles, thus ensuring the operation in steady state.

Fourier analysis is done only after a transient analysis; therefore a .TRAN must be specified before the .FOUR command. The results of Fourier analysis are automatically printed to the output; therefore there is no need to use .PLOT, .PRINT, .PROBE commands. A Spice simulation [8, 9, 10] of the sawtooth function is shown in the following Figure 2.

![Figure 2. Sawtooth waveform in the Probe of OrCad-Pspice.](image)
The VPULSE voltage generator was used to generate the waveform having the following values of the parameters: V1=0, V2=1, TD=0, TR=1, TF=0, PW=1, and PER=1 as shown in Figure 3.

![Figure 3. Sawtooth Spice based source using OrCAD PSpice implementation.](image)

The Fourier analysis is done at the end of the transient analysis and for one period, from final time value-fundamental period to final time value. At the end of the analysis, Spice evaluates the DC component and the amplitudes of the fundamental and up to nine harmonics. It is a good idea to run the analysis for a few periods to ensure the purging of transients and that the system is in a steady state.

The Fourier series components are calculated by the simulator and included in the output file. The arithmetic value of the first ten components of the sawtooth wave and for the arithmetic values of the parameters of the source are provided in the output file of the simulation and shown in Figure 4. This analysis decomposes a waveform into a DC component, a fundamental and harmonics. The fundamental and harmonics reported by Spice simulator can be up to ten. The type of analysis implement by Spice is the Discrete Fourier Transform (DFT). The syntax of the Fourier analysis has the form:

```
.FOUR <frequency> <Output>
```

Where:  
- `<frequency>` is the fundamental frequency, $1/T$ of the waveform to be analyzed, and  
- `<output>` is the node for which the Fourier analysis is made and results are reported.

The input waveform can be reconstructed and expressed in terms of the Fourier series coefficients given in Figure 4. The DC component along the first three harmonics give,

$$u_{in}(t) = 0.495 + 0.318 \sin (\pi t + 178^\circ) + 0.159 \sin (3\pi t + 176^\circ) + 0.106 \sin (5\pi t + 175^\circ) + \ldots$$  \hspace{1cm} (17)

This compares with the exact value of the analytical derivation by Eq. (15) as,

$$u_{in}(t) = 0.5 - 0.318 \sin (2\pi t) + 0.159 \sin (4\pi t) + 0.106 \sin (6\pi t) + \ldots$$  \hspace{1cm} (18)

Examination of the Fourier coefficients or Fourier components as are referred in the output file of OrCAD Spice shown in Figure 4, shows that a major difference occurs in the sign and the phase angle of each one of the components. But observing that the calculated angle is negative and close to 180 degrees, approximating the angle with 180 degrees, and using the trigonometric identity, $\sin (\phi - \pi) = -\sin (\phi)$, the required negative sign is produced. The difference between the exact value of 180 degrees and the values provided by Spice is likely due to the inaccuracies of describing the sawtooth wave and the numerical approximations inherent in the Spice algorithms. An example of inaccurate description of the waveform is the fall time which is finite and not zero, as in the ideal case. As a result, the slope is not infinite but finite.
FOURIER COMPONENTS OF TRANSIENT RESPONSE $V(V_{Vin})$

**DC COMPONENT** = 4.960487E-01

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<th>FREQUENCY (HZ)</th>
<th>FOURIER COMPONENT</th>
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<th>PHASE (DEG)</th>
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Figure 4. First ten components of the source sawtooth voltage as provided in the output file of OrCAD Spice.

**FOURIER ANALYSIS AND SYNTHESIS – SYNTHESIS USING A MATHEMATICAL NOTEPAD**

A free mathematical notepad (SMath) has been introduced in the classroom and used in various presentations by the author. In this case it is used to synthesize a sawtooth waveform from its components. The general form of the Fourier series expansion of a sawtooth waveform was shown in eq. (15), is

$$f(t) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\pi t)$$

(19)

For practical applications, the number of components has to be finite. Here is the advantage of using a computer aided instruction and specifically a mathematical notepad that has the ability to plot the functions. Any specific component can be plotted individually, any number of components can be plotted together, and the sum of any number of components can be added together. Figure 5 displays the DC and first harmonic.

Figure 5. Plot of the average or DC component and first harmonic.
Figure 6. Sum of harmonics for various number of components of a sawtooth waveform.

The plot of the sum of a number of components in Figure 6 clearly displays the ringing due to the Gibbs phenomenon. The phenomenon was first observed by Henry Wilbraham [11]. In 1898, Albert Michelson developed a device that could compute and synthesize the Fourier series [12]. J Willard Gibbs published a paper in 1898 [13] and a correction in 1899 in which he described the overshoot at the point of discontinuity of a function [14].

Figure 7. Comparison of the reconstructed input waveform and the DC component along the first four AC components.
Figure 7 displays and compares the DC component and the first four harmonics along with their sum. Figure 8 displays the sum of the DC component along with the first four harmonics and the approximately ideal function of the first fifty terms of the series.

FOURIER ANALYSIS AND SYNTHESIS – APPLICATION TO A FIRST ORDER SYSTEM

The response of a system to a sawtooth waveform will be considered. The system will be consisted of a passive RC filter. An RC low pass filter is simulated and Fourier analysis is performed to determine the response of the system. The Spice schematic circuit of the system (RC components) and the excitation (Vin source) is simulated. The Schematics graphical entry used and it is shown in Figure 9.

The input and output waveforms are shown in Figure 10. The input waveform is a sawtooth as have been described earlier. The output waveform approximates the input. The output waveform does not reach the zero value. This is due to the charging time constant of the RC network. The output waveform reaches the steady state approximately after one period. Observation of the output waveform indications nearly identical repletion of the waveform after the first period. The Fourier components of the output waveform are available at the out file of the simulation and are shown in Figure 11. Based on the provided values, the Fourier series expression of the output voltage can be constructed as

$$u_o(t) = 0.4998 + 0.270\sin(\pi t - 160^\circ) + 0.099\sin(3\pi t - 154^\circ) + 0.050\sin(5\pi t - 157^\circ) + \ldots \quad (20)$$
There is a difference in the phase of the sinusoidal. The coefficients generated by Spice are positive where the analytical are negative. Taking into account the phase angle in Spice and noting that \(\sin(\alpha - 180^\circ) = -\sin \alpha\) and the expressions are in close agreement. The minor difference between the phase angles \((-178^\circ, -176^\circ, -175^\circ)\) with respect to the expected value of 180° is likely due to the arithmetic approximations of the Spice algorithms.

**Figure 10.** Probe displays of input and output waveforms.

**FOURIER COMPONENTS OF TRANSIENT RESPONSE V(t00146)**

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**Figure 11.** Fourier components of the output waveform.

**FOURIER ANALYSIS AND SYNTHESIS - FIRST ORDER SYSTEM: FOURIER TRANSFORM ANALYSIS**

The frequency components of the input and output waveforms can be generated in the Probe environment. PROBE has a Fourier transform function which can be invoked using the .PROBE command. Figure 9 shows the input and output frequency components of the input and output waveforms. The Fourier transforms results in the amplitude of the components as function of frequency [15, 16, 17]. The type of transform is also a DFT. This is a special implementation of the Fast Fourier Transform (FFT). The resolution of the waveform determines the width of the output spices. The resolution is the reciprocal of the period of the waveform.

**Figure 12.** Probe Fourier transform of the input and output waveforms.
FOURIER ANALYSIS AND SYNTHESIS – FURTHER WORK

Work is under way to extend the results to other waveforms and systems taking advantage of analytical and simulation resources. A number of experiments in an analytical, simulation, and prototyping setup is under preparation to study and verify and evaluate the behavior of specific systems and excitations.

REFERENCES

[7] University of California Berkley SPICE. Electronics Laboratory.

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