Implications of Unchecked Exponential Growth

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***Abstract*—Treatments of applications of the exponential function in Calculus textbooks are often cursory in that they treat unchecked (Malthusian) exponential growth processes without examination of their implications and consequences. Through the use of problems posed to students, we illustrate the implications of such processes when applied to the growth of populations and to the growth in the rates of consumption of nonrenewable resources.**

***Index Terms*—Exponential function, doubling time, population growth, resource consumption.**

I. Introduction

The study of the natural exponential function e^{kt} and its application to various growth and decay phenomena have become standard features of courses in Calculus. Although textbooks typically include applications to the growth of populations and to continuously compounded interest accounts, such treatments are often perfunctory and do not display the ultimate outcomes of unchecked exponential growth [1]. A more thorough analysis is well within student capabilities and we present here several problems, often assigned as projects, to extend the students' understanding of the effects and consequences of such growth. In particular we examine applications to population growth and to growth in the consumption rate of nonrenewable natural resources. These examples were inspired by lectures and writings of the late Albert A. Bartlett [2],

II. Population Growth

The instantaneous size of a population undergoing Malthusian growth is given by

$$N(t) = N_0 e^{kt} \tag{1}$$

in which N_0 is the population size at time $t = 0$ and k is a positive constant analogous to the interest rate in a compound interest problem. A characteristic measure of the rate of growth is the doubling time, T_2, given by

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$$T_2 = \frac{Ln2}{k} \approx \frac{70}{k\%}, \tag{2}$$

in which the second form, often known as the banker's law of 70, requires k to be expressed as a percent. If k remains constant at all times in the process, then so also does the value of T_2

Although these concepts are readily understood, the implications and consequences of exponential population growth are not easily appreciated. It is therefore instructive to pose questions involving computation of doubling times of human populations and we outline two such exercises below. Each assumes growth of the form of (1) under conditions of the current World growth rate of 1.1%/yr [3], corresponding to a doubling time of 63 years.

Question (1): In how many doubling times will the density of human beings on Earth approach one person per square meter of solid land surface? This requires solution of the equation

$$2^n \rho_0 = \rho_n \tag{3}$$

in which ρ_0 is the current density and ρ_n is that after n doublings. Solving (3) for n yields

$$n = Log_2 \frac{\rho_n}{\rho_0}. \tag{4}$$

Taking the current population as $\approx 7\times10^9$ [3] and the World land surface area as $\approx 1.5\times10^{14}$ m^2 [4] results in $\rho_0 \approx 5\times10^{-5}$ m^{-2}. Then setting $\rho_n = 1$ m^{-2} gives $n \approx 14$ doublings which corresponds to ≈ 880 years. Students readily agree that they would feel crowded long before this condition is attained.

Question (2): In how many doubling times will the mass of human beings on Earth equal that of the Earth itself? Here the equations to be solved are analogous to (3) and (4) but with densities replaced by masses. Thus

$$n = Log_2 \frac{M}{m} \tag{5}$$

in which M is the mass of the Earth $\approx 6.0 \times10^{24}$ kg and m is the current mass of human beings. For the latter an average mass of 65 kg [5] results in $m \approx 4.4 \times 10^{11}$ kg; then $n \approx 44$, corresponding to ≈ 2770 years. To our knowledge, no one has yet determined the gravitational and other consequences of having half the planet's mass as a mobile mass located on its surface.

The realization of the nearness in time of these hypothetical events is invariably quite dramatic and causes students to question the assumptions underlying these computations, as well they might. In particular, the constancy of k and the validity of growth in the form of (1) are brought into question. Concerning the value of k, one may take some comfort upon learning that the growth rate has been steadily declining since its peak of 2.19%/yr in 1963 [3].

At this point one can introduce a more realistic growth model such as given by the logistic function

$$N(t) = \frac{M}{1 + Ae^{-at}} \tag{6}$$

in which A and a are positive constants and the quantity M represents an asymptotic upper limit to population size. Moreover, it is easily shown that at early stages in the growth this equation reduces to the form of (1), suggesting that a population tends to follow the Malthusian form as long as it does not sense an upper limit to its size. Subsequently there occurs a point of inflection in the growth curve when the population size is one-half the ultimate size and students find it interesting to speculate on the cause of this change in behavior. Finally, it may be expected that students will appreciate a population model containing a growth-limiting feature.

III. RESOURCE CONSUMPTION

For the consumption rate of a resource, we again consider the form of (1),

$$r(t) = r_0 e^{kt} \tag{7}$$

where $r(t)$ is the consumption rate at time t, r_0 is that at an initial time $t = 0$ and k is again a positive constant with units of reciprocal time. The total resource, R, consumed in a time interval $[t_1, t_2]$ is obtained by integrating this function; thus

$$\begin{aligned} R &= \int_{t_1}^{t_2} r(t)dt \\ &= \frac{r_0}{k}\left(e^{kt_2} - e^{kt_1}\right). \end{aligned} \tag{8}$$

Here T_2 as given in (2) represents the time required for a doubling in the consumption rate, $r(t)$. If the resource under consideration is oil, the units of the consumption rate can be barrels per year (bbls/yr) and those of R and T_2 respectively would be barrels and years. Again we pose several problems to illustrate the effects of unchecked exponential growth.

Question (3): Show that, under conditions of (7), in any doubling interval the amount of the resource consumed exceeds or is approximately the same as has been consumed in all of history up to the start of the interval. This statement is almost always greeted with disbelief but is easily proved. Consider that consumption of the resource commenced at time $t = 0$. Then the amount consumed between then and time $t = t_1$ is

$$R_1 = \frac{r_0}{k}(e^{kt_1} - 1). \tag{9}$$

Next we compute the integral for time interval $[t_1, t_1 + T_2]$ to obtain

$$\begin{aligned} R_2 &= \frac{r_0 e^{kt_1}}{k}(e^{kT_2} - 1) \\ &= \frac{r_0 e^{kt_1}}{k}(e^{Ln2} - 1) \\ &= \frac{r_0 e^{kt_1}}{k} \geq R_1. \end{aligned} \tag{10}$$

In (10) the equality holds when, for reasonable values of the exponent in (9), $e^{kt_1} \gg 1$.

At this point it is useful to introduce a real example. During much of the 20th century, the World-wide consumption rate of oil increased at an average annual rate of 7%/yr, corresponding to a doubling time of 10 yrs [6]. Thus, for example, in the decade of the 40s, there was as much oil consumed as had been consumed in all of history prior to 1940 and again in the decade of the 1950s there was as much oil consumed as had been consumed in all of history prior to 1950 and in the decade of the 1960s there was as much oil consumed as had been consumed in all of history prior to 1960, and so on. This pattern persisted until political events in the 1970s slowed the consumption of oil [7], [8]. Understand, the 7%/yr refers not to consumption of the resource but to the **rate** of consumption. This being understood, one should now ask how long can a non-renewable resource such as oil last under an exponentially increasing consumption rate. This prompts the next question.

Question (4): Compute the lifetime of a non-renewable resource undergoing an exponentially increasing rate of consumption. First we calculate a general expression for the lifetime T of a resource of finite size R undergoing consumption at a rate given by (7). If we represent the present time by $t = 0$, we must solve

$$\begin{aligned} R &= r_0 \int_0^T e^{kt} dt \\ &= \frac{r_0}{k}(e^{kT} - 1). \end{aligned} \tag{11}$$

for the unknown lifetime *T*. There results

$$T = \frac{1}{k} Ln\left(1 + \frac{kR}{r_0}\right). \qquad (12)$$

Note that if this resource were undergoing consumption at a **constant** rate, say r_0, its lifetime would be R / r_0.

Next we insert numerical values and compare the lifetime for the case of consumption at a constant rate with those for exponentially increasing rates. Suppose we consider the case of oil and take an extreme case for the size of *R*. We can state with good precision an absolute upper limit to the amount of oil in the Earth, namely consider the Earth as one giant oil tank. Thus modeling the Earth as a sphere of radius 4,000 miles and converting the volume to barrels yields $R \approx 10^{21}$ bbls. At the present consumption rate of $\approx 10^{10}$ bbls/yr, this resource would last $\approx 10^{11}$ yrs, i.e., longer than the Earth will be in existence [9]. Now examine the reduction in the life of this resource under an exponentially increasing rate of 7%/yr. The result from (12) is ≈ 300 yrs!

How would the result differ if the exponentially increasing consumption rate were reduced to say 2%/yr? That would indeed lengthen the lifetime but only to ≈ 1070 yrs. But consider further that we do not possess an Earth full of oil. Then under realistic estimates of oil reserves, an exponentially increasing consumption rate would deplete the resource much sooner. Since new sources are frequently found, estimates of proven oil reserves constitute a moving target but for purposes of computation one might use a 2012 estimate of $\approx 1.4 \times 10^{12}$ bbls [10]. We leave it to the reader to compute the lifetime of this resource under conditions both of a constant rate and of exponentially increasing rates of consumption. Be assured, the results will stimulate interest in our energy policy.

IV. CONCLUSION

It has been our purpose to demonstrate the implications of unlimited exponential growth and it is our hope that they will be understood by leaders in the fields of technology, business and government. Unfortunately that is not the case when one hears forecasts of the lifetime of natural resources which falsely assume consumption to be at a constant rate [11].

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[6 A. A. Bartlett, *Forgotten fundamentals of the energy crisis*, Am. J. Phys. **46**, 887 (1978).

[7] http://en.wikipedia.org/wiki/1973_oil_crisis.

[8] http://en.wikipedia.org/wiki/1970s_energy_crisis.

[9] In $\approx 5\times10^9$ years the Sun will enter its Red Giant phase and thereby envelope the Earth.

[10] http://en.wikipedia.org/wiki/Oil_reserves.

[11] See [2] and [6] for numerous examples.