

Statistical Programming Education Through Physics Research

Daniel Katz

Department of Physics, University of Massachusetts Lowell

Lowell, MA 01852, USA

daniel_katz1@student.uml.edu

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Abstract

General Relativity (GR) is currently our best guess at how gravity works and is one of the most successful physical theories ever enunciated. However, aside from being completely incompatible with Quantum Field Theory (QFT), which is also astonishingly accurate, GR forces us to accept some strange consequences when applied to the kinematics of galaxies. GR can only accurately describe the rotation of spiral galaxies if there is a huge halo of invisible “dark matter” around said galaxies. An open problem, to which there are many approaches in theoretical physics right now is reconciling GR and QFT into a single consistent theory. In this paper we examine how one such candidate theory, called Yang-Mills Gravity (YMG), handles the galactic rotation problem. Doing so requires a lot of number crunching with big tables of astronomical data. Many of the individual parts of the calculation are essentially introductory programming problems like linear regression or constrained optimization. A group of students from Gardner High School (Gardner, MA) are learning to use the R programming language (and in the process some physics) by building up and combining these small subroutines.

1 Introduction

For the reader’s convenience, we will briefly describe the conflict between GR and QFT in order to motivate the introduction of YMG. For more details on either, see [1, 2]. There are many subtle logical arguments that show this conflict, but just the combination of GR’s and QFT’s postulates is enough to generate a contradiction. According to GR, gravity is not a force at all. Rather, the presence of matter and energy distorts space-time which then modifies the trajectories of objects in it. The distortion of space-time is quantified by the metric tensor $g^{\mu\nu}$, which can be defined by its effect on the line element, *viz.*

$$ds^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g^{\mu\nu} dx_\nu dx_\mu \quad (1)$$

where the zero index corresponds to the time component of four-dimensional space-time. Thus, given the metric tensor we can calculate such things as trajectories of objects in a gravitational field. Einstein’s field equations describe exactly how the four-dimensional space-time metric changes in response to matter and energy:

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \quad (2)$$

The tensor $G^{\mu\nu}$ is the so-called Einstein tensor and is a complicated deterministic function of the metric $g^{\mu\nu}$.

Meanwhile the energy-momentum tensor, $T^{\mu\nu}$, encodes all information about the matter, energy, and momentum in the space. The coefficient of the energy-momentum tensor is chosen so that, in the low-speed low-gravity limit, Einstein’s field equations reduce down to Newton’s law of gravity. The specific forms of these tensors are not relevant to this discussion. In curved four-dimensional space-time, Newton’s second law reads

$$g^{\mu\nu} \frac{\partial S}{\partial x_\mu} \frac{\partial S}{\partial x_\nu} - m^2 = 0 \quad (3)$$

where S is the trajectory we wish to solve for and m is the mass of the object following that trajectory.

QFT is dominated by its algebra. According to quantum mechanics, observables like position and momentum can no longer be represented by variables as they are in classical mechanics. Rather, they need to be promoted to Hermitian operators which act on the space of all possible configurations of the system. QFT takes this one step further by making entire fields into so-called field operators. Two field operators ψ and ϕ are determined by

$$[\psi(t, \vec{x}), \phi(t, \vec{y})] = \delta(\vec{x} - \vec{y}) \quad (4)$$

where $[A, B] \equiv AB - BA$ is the commutator of two operators, and $\delta(\cdot)$ is Dirac’s delta distribution. Eqn. 4 implies that if ψ and ϕ are not in the same place at the same time they cannot interact. This relation, and generalizations of it just enforce a sense of causality in QFT.

This procedure of promoting field functions into field operators is called “second quantization” and it leads to all of the success of the Standard Model of particle physics. But what happens if we try to second quantize GR? Essentially, this is a question of what would happen if space-time geometry were subject to quantum fuzziness. We need to be able to write out Eqn. 4 explicitly for the “quantum metric tensor” in order for it to help us determine the field operators’ algebra. The problem is that, in order to write it out explicitly we need to know the distance between the two fields. To know the distance, we need to know the quantum metric. But we were trying to write out Eqn. 4 in order to solve for the quantum metric in the first place! This puts us in the unenviable position of needing to already know the quantum metric before we set out to solve for it.

It should be emphasized that the above argument is just one of many which show that both GR and QFT cannot be completely correct in their current forms. A more detailed account of the tension between physics’ two greatest theories can be found in [3, 4].

YMG seeks to approach gravity from a QFT perspective [5]. To incorporate a type of interaction into a physical system in QFT, one need only make the system’s Lagrangian invariant under certain mathematical transformations. New fields which mediate the interaction then follow from the Lagrangian’s modified form. For example, all physical electromagnetic properties and phenomena are unchanged by a global shift in the electrical potential energy. Formally this is invariance under the $U(1)$ group, and it leads to both the existence of the electromagnetic field and to conservation of electric charge. In the 1950’s C.N. Yang and R. Mills came up with a general procedure for working out the field and its interactions associated with a given symmetry [6]. The details of this method are beyond the scope of this paper but the interested reader is referred to [1] for a reasonable introduction. In YMG we start by assuming local translational symmetry (formally, invariance under the T_4 group) and use the Yang-Mills machinery to find the corresponding field, in this case a tensor we’ll call $\phi^{\mu\nu}$, and its equations of motion. In the classical limit, where the action of the system is much larger than the fundamental quantum of action, (*i.e.* $\hbar \rightarrow 0$) the equations motion for $\phi^{\mu\nu}$ always take the form

$$G^{\mu\nu} \frac{\partial S}{\partial x_\mu} \frac{\partial S}{\partial x_\nu} - m^2 = 0 \quad (5)$$

$$G^{\mu\nu} \equiv (1 + \phi^{\mu\sigma})(1 + \phi_{\sigma\nu}) \quad (6)$$

This equation also works for massless fields by setting $m = 0$. Now we get to see why YMG is considered to be a theory of gravity. Eqn. 5 looks and behaves just like Eqn. 3 with $G^{\mu\nu}$ playing the role of an effective metric tensor. It is merely an *effective* metric because we started its derivation by assuming flat space-time. Thus, YMG explains gravity as the macroscopic effect of the quantum field corresponding to the T_4 symmetry group.

Armed with equations of motion for an effective metric tensor we can make all the same sort of calculations which one normally makes with GR. The canonical phenomena used to verify GR, perihelion precession of Mercury and the bending of light around massive objects, are also predicted to within the limit of experimental accuracy by YMG [5, 7, 8].

2 Solution for a Spherically Symmetric Source

Very often in astrophysics we would like to use a known form for the gravitational potential to infer the properties of objects from their observed motions. A common scenario is that of one object orbiting another much heavier spherically symmetric body. We can solve for either GR’s metric tensor or YMG’s effective metric by using $T^{00} = -M\delta(\vec{r})$ as a reasonable approximation to a spherically symmetric source (this approximation is perfectly fine as long as we’re outside the source). In the case of GR the equations can be solved exactly giving the famous Schwarzschild line element:

$$ds^2 = - \left(1 + \frac{GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (7)$$

The equations relating the components of YMG’s effective metric are very complicated and non-linear so a nice closed form like the Schwarzschild metric cannot be written down. However, we can use successive approximations to solve for the effective metric components to whatever order desired. Solving to order $1/r^2$ leads to the results in [5, 7]. It is often easier to perform calculations in a Newtonian setting by extracting an effective gravitational potential from the (effective or real) metric. It is relatively straightforward to show [2] how the effective potential should be related to the various metric components. Via a computer algebra system we have recently shown that there is a first-order solution to the equations of motion for the effective metric with a spherically symmetric source such that the effective potential takes the form

$$\phi_{eff} = \beta r + \frac{GM(1 + GM\beta)}{r(1 + 2GM\beta)} \quad (8)$$

where G is Newton’s constant divided by an implicit factor of c^2 ¹, M is the mass of the source, and β is a free parameter of the theory. Let us investigate a consequence of the linear-in- r term in the effective potential.

¹It is quite common to chose units such that c , the speed of light, and \hbar , the reduced Planck’s constant, are both dimensionless and equal to 1.

3 Rotation Curves & Dark Matter

Consider a galaxy with matter density $\rho(\vec{r})$ and mass $M = \int \rho(\vec{r})d^3\vec{r}$. If $\phi(\vec{r})$ is the gravitational potential caused by the galaxy then the rotational speed of galactic matter at a radial distance r from the center of the galaxy is

$$v(r) = \sqrt{r \frac{d}{dr} \int \rho(\vec{r}')\phi(|\vec{r} - \vec{r}'|)d^3\vec{r}'} \quad (9)$$

On the scale of galaxies, matter is sufficiently dilute that Newtonian approximations to mechanics work just fine. This is the justification for using the Newtonian Eqn. 9 with the effective potential. Comparing the results of Eqn. 9 using GR's effective potential with astronomical observations yields a disturbing discovery.

This figure shows the typical qualitative situation; the agreement is actually pretty good near the center of the galaxy, but farther out it diverges from observations. This result follows from constructing ρ by fitting observed luminosity profiles, but the discrepancy implies that either ρ or ϕ is wrong. Historically, because of its successes, people have been uneasy about the possibility that the effective potential, and hence GR, is wrong. Rather, it is common to assume that ρ is composed out of the matter we can see plus a contribution from matter we can't; dark matter. In spite of its popularity, dark matter is not the only feasible solution to the galactic rotation curve problem described above.

As has been known for some time by Modified Newtonian Dynamics (MOND) theorists, another way to get rotation curves which don't sag at distant radii is to introduce a linear term in the gravitational potential [9]. This was the motivation for searching for a solution to YMG's equations of motion of the effective metric which would lead to a linear term in the potential. Now, the coefficient of the linear term, β is to be determined from experimental data.

4 Data Analysis & Programming Education

To estimate β for a particular galaxy, we need to know the radial matter distribution of that galaxy along with its rotation curve. The latter is determined by measuring the Doppler shift of the $H1$ and/or $H\alpha$ emission lines from hydrogen in the galaxy. The matter distribution is much more difficult to determine because it depends on parameters, such as the estimated mass of the galaxy and its mass-to-light ratios across various bands, which can only be estimated from models and not measured directly. For raw astronomical data we go to the Sloan Digital Sky Survey (SDSS), a massive CCD array survey of objects in the night sky. The SDSS is ideal not only because of its bulk of data (around 70 terabytes in all), but because all of that data is free to the public. In addition to raw data the SDSS offers many other data products,

including calculations under various models of all the parameters we need to construct the radial matter density. The construction of these density functions is relatively straightforward, so the task is good for training beginning programmers. The actual data for a galaxy's luminosity profile takes the form of binned azimuthally average fluxes over five different bands in the visible and IR part of the electromagnetic spectrum. We weight each band with the estimated mass-to-light ratios and fit a single parameter function to the result. Traditionally, astronomers like to use the so-called Sérsic profile to fit galactic luminosity profiles:

$$\rho(r) \propto e^{-x^\alpha} \quad (10)$$

where $x = r/R_0$, R_0 is the half-light radius of the galaxy, and α is a parameter to be tuned to each particular galaxy. The Sérsic fit is one-dimensional in the radial direction; we are assuming that the galactic disk is azimuthally symmetric and that it is much wider than it is thick. It fits a variety of galaxy types pretty well [10] but is quite unruly as part of an integrand. A few years ago Spergel showed that

$$\rho(r) \propto \left(\frac{x}{2}\right)^\nu \frac{K_\nu(x)}{\Gamma(\nu+1)} \quad (11)$$

fits on average just as well as the Sérsic profile [11]. Here ν is the tunable parameter, $\Gamma(\cdot)$ is the Gamma function, and $K_\nu(\cdot)$ is a modified Bessel function of the second kind. Realistically, the difference between the sum of residuals after fitting data with a Sérsic versus a Spergel profile is going to be small compared to measurement uncertainties. The real reason to use the Spergel profile is that it allows the integral in Eqn. 9 to be carried out exactly. This is because the expansion of $1/|\vec{r} - \vec{r}'|$ in terms of Bessel functions of the first kind, $J_m(\cdot)$, *givenby*

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk J_m(kr) J_m(kr') e^{im(\phi - \phi') - k|z - z'|} \quad (12)$$

lets us take advantage of the solubility of integrals of multiple Bessel functions. The predicted velocity from Eqn. 9 is the square root of a combination of powers of $x = r/R_0$, β , and hypergeometric functions.

The parameter β wouldn't be very interesting if it needed to be tuned to each individual galaxy. Instead, we seek a universal value which best fits a large sample of galaxies. The galaxies in any sample, no matter how big, are bound to have their peculiarities. Computing the half-light radius R_0 and the best fitting ν in a Spergel profile for each galaxy allows those parameters to absorb some of the "locality" out of our estimate for β , making it more universal. Turning data tables from SDSS into (R_0, ν) pairs for each galaxy is a multi-step process involving regression, spline interpolation, and numerical differentiation.

Through the NSF's GK-12: Vibes & Waves Fellowship the author visits 11th grade physics classes at Gardner

High School to augment their physics classroom experience. A small group of these students is involved in an after-school R language programming seminar run by the author. The students are all beginner programmers, so the training includes simple example programs based on what was covered in their class that day. For example, we could write a loop to perform the same operation on many elements (such as the students have to do when filling out the tables in their physics lab reports) or we could use R's element-wise vector operations. The students have progressed nicely and are presently able to write their own linear regression code. The task of finding (R_0, ν) pairs for each galaxy will be a challenging and educational project for the students.

if it were based on observations of only six galaxies. Thus, the process of data collection/selection continues. Meanwhile, the student programming talent grows as they put together a poster for an upcoming research symposium at the University of Massachusetts Lowell. If the best fitting β for a reasonable sample size is not a good fit at all, then one would have to question the validity of the linear term in the effective gravitational potential of YMG. Otherwise, we will have shown that YMG doesn't need dark matter to correctly predict galactic rotation curves. Once the value of β is known, the consequences of $\phi = \beta r \dots$ beyond rotation curves will have to be checked for consistency with observations.

5 Outlook & Conclusion

While the theoretical brickwork has been laid, we presently have complete data for only half a dozen galaxies. Complete data for even one galaxy would be enough for our students to test their algorithms, but it would be something of a stretch to claim any value of β as universal

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2

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